# Vortex definitions and bifurcations of vortical structures Part 2 — Vorticity and the Q-criterion

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## Overview

### 1) V

- Vorticity
- Point vortices
- Vortex definition # 2
- The onset of vortex dynamics in the cylinder wake
- 2 Vortex definition # 3: The Q-criterion
  - Q-vortices in boundary layer eruption



### Vorticity in two dimensions

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Navier-Stokes equation

$$rac{D \mathbf{v}}{D t} = -rac{1}{
ho} 
abla 
ho + 
u \Delta \mathbf{v}, \qquad rac{D}{D t} = rac{\partial}{\partial t} + \mathbf{v} \cdot 
abla.$$

Take the curl — vorticity transport equation

$$\frac{D\omega}{Dt} = \nu \Delta \omega$$

Ideal fluids ( $\nu = 0$ )

Vorticity is frozen in the fluid.

A *Point vortex* of circulation  $\Gamma$  centered at  $\mathbf{x}_0$ ,  $\omega = \Gamma/(2\pi)\delta(\mathbf{x} - \mathbf{x}_0)$  induces a velocity field

$$\mathbf{v} = rac{\mathsf{F}}{2\pi} rac{\widehat{\mathbf{x} - \mathbf{x}_0}}{|\mathbf{x} - \mathbf{x}_0|^2}$$

### Ideal fluids cont.

N point vortices placed at  $\mathbf{x}_{\alpha}, \alpha = 1, \dots, N$  induce a velocity field

$$\mathbf{v} = \sum_{lpha=1}^{N} rac{\Gamma_{lpha}}{2\pi} rac{\widehat{\mathbf{x}-\mathbf{x}_{lpha}}}{|\mathbf{x}-\mathbf{x}_{lpha}|^2}$$

Each vortex is a material point and moves in the velocity field from the other vortices

$$\frac{d\mathbf{x}_{\beta}}{dt} = \sum_{\substack{\alpha=1\\ \alpha\neq\beta}}^{N} \frac{\Gamma_{\alpha}}{2\pi} \frac{\widehat{\mathbf{x}_{\beta} - \mathbf{x}_{\alpha}}}{|\mathbf{x}_{\beta} - \mathbf{x}_{\alpha}|^2}, \quad \beta = 1, \dots, N.$$

Point vortices successfully used to model cylinder wakes (von Kármán, 1912)



#### How to generalize to real viscous flows where vorticity diffuses, $\frac{D\omega}{Dt} = \nu \Delta \omega$

## Vortex definition #2

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Vorticity is Galilean invariant.

### Generalizing point vortices

A feature point of a vortex is a local extremum of vorticity,

$$\partial_x \omega = \partial_y \omega = 0,$$

$$\det(H)>0, \quad H=egin{pmatrix} \partial_{xx}\omega & \partial_{xy}\omega\ \partial_{xy}\omega & \partial_{yy}\omega \end{pmatrix}$$

If det(H) < 0, the critical point of  $\omega$  is a saddle.



### Motion of critical points of vorticity (extrema and saddles)

A critical point (x(t), y(t)) of vorticity fulfills

$$\partial_x \omega(x(t),y(t),t)=0, \;\; \partial_y \omega(x(t),y(t),t)=0$$

Implicit differentiation yields equations of motion

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -H^{-1} \begin{pmatrix} \partial_{xt}\omega \\ \partial_{yt}\omega \end{pmatrix} = \begin{pmatrix} \frac{\partial_{xy}\omega\partial_{yt}\omega - \partial_{yy}\omega\partial_{xt}\omega \\ \det(H) \\ \frac{\partial_{xy}\omega\partial_{xt}\omega - \partial_{xx}\omega\partial_{yt}\omega }{\det(H)} \end{pmatrix}$$

Vortices are created or destroyed when det(H) = 0

### Cusp or saddle-center bifurcation of vortices

**Theorem** Assume the Hessian *H* has zero as a simple eigenvalue at a critical point at (x,y,t) = (0,0,0), and choose the coordinate system such that

$$H(0,0,0)=H_0=egin{pmatrix} 0&0\0&\partial_{yy}\omega_0 \end{pmatrix}$$

Assume the non-degeneracy conditions

$$A = \partial_{yy}\omega_0 \neq 0, \quad B = \partial_{xt}\omega_0 \neq 0, \quad C = \partial_{xxx}\omega_0 \neq 0.$$

Then there are critical points of vorticity given by

$$x(t)=\pm\sqrt{-rac{2B}{C}t}+\mathcal{O}(t), \hspace{1em} y(t)=\left(-rac{1}{A}\partial_{yt}\omega_{0}+rac{B}{AC}\partial_{xxy}\omega_{0}
ight)t+\mathcal{O}(t^{3/2})$$

If B/C > 0 the two critical points exist for t < 0 and merge and disappear at the origin at t = 0. If B/C < 0 the points are created at t = 0 and exist for t > 0. In both cases, one of the critical points is a saddle, the other is an extremum.

## Cusp bifurcation example

$$\omega = t(y-x) + y^2 - \frac{1}{3}x^3, \qquad \partial_x \omega = -t - x^2, \quad \partial_y \omega = t + 2y.$$
  
Critical points

$$x = \pm \sqrt{-t}, \quad y = -\frac{1}{2}t.$$

Easy to check that the assumptions of the theorem are fulfilled at (x,y,t) = (0,0,0).

### Wake of cylinder close to wall

Rasmus Ellebæk Christiansen, Master's Thesis, 2013.

### The role of the vorticity transport equation

$$\partial_t \omega + (\mathbf{v} \cdot 
abla) \omega = 
u \Delta \omega$$

In the regular case, the equations of motion for the critical points of vorticity become

$$\begin{split} \dot{x} &= u - \nu \frac{\partial_{yy} \omega \Delta \partial_x \omega - \partial_{xy} \omega \Delta \partial_y \omega}{\det(H)} \\ \dot{y} &= v - \nu \frac{\partial_{xx} \omega \Delta \partial_y \omega - \partial_{xy} \omega \Delta \partial_x \omega}{\det(H)} \end{split}$$

For the cusp bifurcation we get

$$egin{aligned} B &= \partial_{xt}\omega_0 = 
u\Delta\partial_x\omega_0, & rac{x(t)^2}{
u t} = -2\left(1+rac{\partial_{xyy}\omega_0}{C}
ight) + \mathcal{O}(t) \ y(t) &= \left(
u + 
urac{\partial_{xxy}\omega_0\partial_{xyy}\omega_0 - \partial_{xxx}\omega_0\partial_{yyy}\omega_0}{A}
ight)t + \mathcal{O}(t^2) \end{aligned}$$

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### The onset of vortex dynamics in the cylinder wake



- Flow is steady and symmetric at modest Reynolds numbers.
- Steady flow becomes unstable at

$$extsf{Re_{ extsf{crit}}D} = rac{U_{ extsf{crit}}D}{
u} pprox$$
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via a symmetry-breaking, super-critical Hopf-bifurcation.

• Instability leads to formation of "Karman vortex street" via periodic shedding of vortices with a characteristic frequency.

## Vorticity field pre- and post-Hopf bifurcation

 Before Hopf bifurcation: Vorticity is generated on no-slip boundaries and then advected downstream; diffusion spreads out the profile as x → ∞. Flow is symmetric about y = 0.



"Carpet plot" of vorticity,  $z = \omega(x,y)$ , above logarithmic colour contours of  $|\omega(x,y)|$ .

### Vorticity field pre- and post-Hopf bifurcation

• After Hopf bifurcation: Time-periodic, asymmetric flow. Vorticity field is advected downstream [Karman vortex street].

"Carpet plot" of vorticity,  $z = \omega(x, y, t)$ , above logarithmic colour contours of  $|\omega(x, y, t)|$ .

## Fluid dynamics near the Hopf bifurcation

Difficult to simulate close to bifurcation due to long transients

Theory: Flow close to the Hopf bifurcation is well approximated by

$$\mathbf{v}(x,y,t;Re) pprox \mathbf{v}(x,y;Re_{ ext{crit}}) + \varepsilon \,\, \widehat{\mathbf{v}}(x,y) \,\, e^{\mathrm{i}\Omega t}$$

where

- $\mathbf{v}(x,y; Re_{crit})$  is the steady flow at the Hopf bifurcation
- $\hat{\mathbf{v}}(x,y)$  is a critical eigenfunction of the linearized problem at the Hopf bifurcation
- $i\Omega$  is the corresponding critical eigenvalue
- $\varepsilon \sim (Re Re_{\rm crit})^{1/2}$  is a proxy for the excess Reynolds number (above the critical value).



• Logarithmic colour contours of  $|\omega(x,y,t)|$ .



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## Location/existence of cusp bifurcation as function of $\varepsilon$

Plot of zero levels of  $\partial \omega / \partial x$  (red),  $\partial \omega / \partial y$  (blue) at time when vortex is created. Location of bifurcation indicated by green marker.



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 Cusp doesn't seem to disappear – it just moves downstream as ε is reduced!

#### Cusp bifurcation "disappears to" infinity as $\varepsilon \to 0$ ??

Plot of cusp position,  $X_{Min}(\varepsilon)$ :



• Observation:

 $X_{\rm Min}\sim arepsilon^{-1/2}$ 

# Well, not quite ...



- No vortices are created when  $\varepsilon < 0.00057$
- For  $\varepsilon = 0.00057$  a vortex is created at  $X_{
  m Min} = 117.1$











- Karman vortex street develops at finite  $0.00054 < \varepsilon < 0.000631$ .
- Cusp bifurcation first appears far downstream of cylinder...



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- Karman vortex street persists for finite length and then disappears via a reverse cusp bifurcation when diffusion of vorticity smoothes out the local maxima.
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### Vortex definition # 3:The *Q*-criterion

The velocity gradient tensor can be decomposed into a symmetric and a skew-symmetric part

$$abla \mathbf{v} = egin{pmatrix} \partial_x u & \partial_y u \ \partial_x v & \partial_y v \end{pmatrix} = \mathbf{S} + \mathbf{\Omega},$$

$$egin{aligned} \mathbf{S} &= rac{1}{2} ( 
abla \mathbf{v} + 
abla \mathbf{v}^{ op} ) = rac{1}{2} egin{pmatrix} 2 \partial_x u & \partial_x v + \partial_y u \ \partial_x v + \partial_y u & 2 \partial_y v \end{pmatrix}, \ \mathbf{\Omega} &= rac{1}{2} ( 
abla \mathbf{v} - 
abla \mathbf{v}^{ op} ) = rac{1}{2} egin{pmatrix} 0 & \omega \ -\omega & 0 \end{pmatrix}. \end{aligned}$$

The Q-criterion: A vortex is a region where rotation dominates shear,

$$egin{aligned} Q &= || m{\Omega} ||^2 - || m{S} ||^2 > 0, \qquad || m{A} ||^2 = \mathrm{tr} (m{A} m{A}^T). \ Q &= \mathrm{det} (
abla m{v}) = \partial_x u \partial_y v - \partial_y u \partial_x v. \end{aligned}$$

Galilean invariant!

### Bifurcation of Q-vortices

Bifurcation occurs at critical points of Q,  $\partial_x Q = \partial_y Q = 0$ . If the Hessian of Q is positive or negative definite, and  $\partial_t Q \neq 0$ , a *punching bifurcation* occurs



If the Hessian of Q is indefinite, and  $\partial_t Q \neq 0$ , a *pinching bifurcation* occurs



# Bifurcation of Q-vortices from a no-slip wall

#### Wall punching



#### Wall pinching



### Q-vortices in boundary layer eruption

### Connection between vorticity and Q



There is no simple general connection between critical points of vorticity and Q-vortices. However:

If the flow has rotational symmetry around an extremum of vorticity, there is a Q-vortex around that point.

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- Local extrema of vorticity are a useful generalization of point vortices for viscous flows
- Equations of motion and a bifurcation theory describing creation and merging of vortices can be derived
- The vortices in the Karman vortex street are created at a Reynolds number slightly higher than the critical value for onset of oscillations, at a distance  $\approx 100$  diameters downstream
- The *Q*-criterion identifies a vortex as a region where vorticity dominates shear
- A bifurcation theory for Q-vortices can be derived